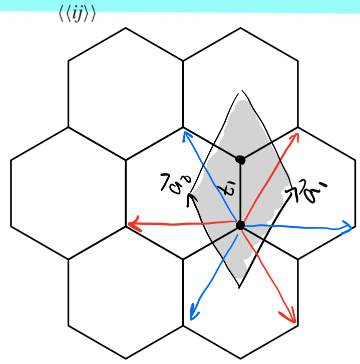
$$H = \Delta \sum_i (-)^{\tau_i} c_i^\dagger c_i + t_1 \sum_{\langle ij \rangle} (c_i^\dagger c_j + \text{h.c.}) + t_2 \sum_{\langle\langle ij \rangle\rangle} (i c_i^\dagger c_j + \text{h.c.}) \qquad (5.1)$$



Rewrite in 1st quantized notation

$\vec{a}_1 = (\sqrt{3}, 0)$, $\vec{a}_2 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $\vec{\tau}_B = (\frac{\sqrt{3}}{2}, \frac{1}{2})$

$\vec{a}_1 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$

$\vec{a}_2 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$

$\vec{\tau}_B = (0, +1)$

$$H = t_1 \sum_{\vec{R}} \Big[|\vec{R} + \cancel{\vec{\tau}_B}, B\rangle\langle \vec{R}, A| + |\vec{R} - \vec{a}_1 + \cancel{\vec{\tau}_B}, B\rangle\langle \vec{R}, A| + |\vec{R} - \vec{a}_2 + \cancel{\vec{\tau}_B}, B\rangle\langle \vec{R}, A| + \text{h.c.} \Big] \longrightarrow H_1$$

$$+ \Delta \sum_{\vec{R}} \Big[|\vec{R}, A\rangle\langle \vec{R}, A| - |\vec{R} + \cancel{\vec{\tau}_B}, B\rangle\langle \vec{R} + \cancel{\vec{\tau}_B}, B| \Big] \longrightarrow H_\Delta$$

$$+ t_2 \sum \Big[i|\vec{R} + \vec{a}_2, A\rangle\langle \vec{R}, A| + i|\vec{R} - \vec{a}_1, A\rangle\langle \vec{R}, A| + i|\vec{R} + \vec{a}_1 - \vec{a}_2, A\rangle\langle \vec{R}, A|$$
$$- i|\vec{R} + \vec{a}_2, B\rangle\langle \vec{R}, B| - i|\vec{R} + \vec{a}_1, B\rangle\langle \vec{R}, B| - i|\vec{R} + \vec{a}_1 - \vec{a}_2, B\rangle\langle \vec{R}, B| \Big] \longrightarrow H_2$$

all still applies

$|B\rangle\langle A| = \sigma^-$, $\quad H_1(\vec{k}) = t_1 (1 + e^{+i\vec{k}\cdot\vec{a}_1} + e^{i\vec{k}\cdot\vec{a}_2}) \sigma^- + \text{h.c.}$

$$H_\Delta(\vec{k}) = \Delta \sigma_z$$

$$H_2 = i t_2 (e^{-i\vec{k}\cdot\vec{a}_2} + e^{i\vec{k}\cdot\vec{a}_1} + e^{-i(\vec{a}_2 - \vec{a}_1)\cdot\vec{k}}) \sigma_z + \text{h.c.}$$
$$= 2t_2 (\sin \vec{k}\cdot\vec{a}_2 - \sin \vec{k}\cdot\vec{a}_1 + \sin \vec{k}\cdot(\vec{a}_2 - \vec{a}_1)) \sigma_z$$

$$H_\Delta(\vec{k}) + H_2(\vec{k}) = \Big[\Delta + 2t_2 [\sin \vec{k}\cdot\vec{a}_2 - \sin \vec{k}\cdot\vec{a}_1 + \sin \vec{k}\cdot(\vec{a}_2 - \vec{a}_1)] \Big] \sigma_z$$

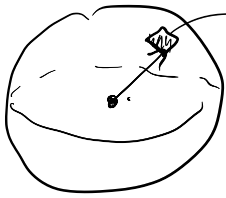
$$H = \Big[t_1 (1 + e^{-i\vec{k}\cdot\vec{a}_1} + e^{-i\vec{k}\cdot\vec{a}_2}) \sigma^+ + \text{h.c.} \Big] + \Big[\Delta + 2t_2 (\sin \vec{k}\cdot\vec{a}_2 - \sin \vec{k}\cdot\vec{a}_1 + \sin \vec{k}\cdot(\vec{a}_2 - \vec{a}_1)) \Big] \sigma_z$$

Let's make sense of this!

(1) $\Delta = 0 = t_2$ — What is the band structure? Graphene!

$H = (\quad) \sigma_x + (\quad) \sigma_y$ — What is Berry curvature?

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\phi(k_x, k_y)} \end{pmatrix} \qquad A_\mu = \langle \psi | i\partial_\mu \psi \rangle = -\frac{1}{2} \partial_\mu \phi$$
$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0$$

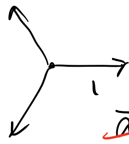


$\Omega_{k_x k_y}\, dk_x\, dk_y$] Needed for Berry curvature (2 bands)

Q: Where is graphene gapless?

$$1 + e^{-i\vec{k}\cdot\vec{a}_1} + e^{-i\vec{k}\cdot\vec{a}_2} = 0$$

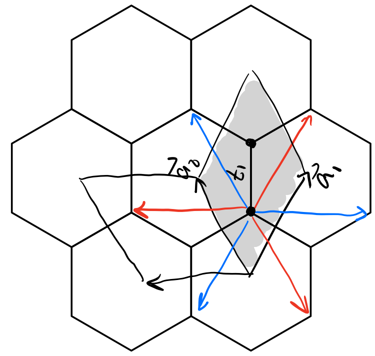
1. Possibility: $\begin{cases} \vec{k}\cdot\vec{a}_1 = \frac{2\pi}{3} \\ \vec{k}\cdot\vec{a}_2 = -\frac{2\pi}{3} \end{cases}$



$\cancel{\vec{a}_1 = (\sqrt{3}, 0)}$

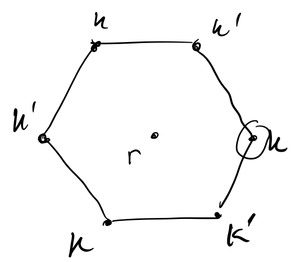
$\vec{a}_1 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$

$\vec{a}_2 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$



$$\frac{\sqrt{3}}{2}k_x + \frac{3}{2}k_y = \frac{2\pi}{3} \qquad k_y = 0$$

$$-\frac{\sqrt{3}}{2}k_x + \frac{3}{2}k_y = -\frac{2\pi}{3} \qquad \sqrt{3}k_x = \frac{4\pi}{3} \Rightarrow k_x = \frac{4\pi}{3\sqrt{3}}$$



$$\vec{K} = \frac{4\pi}{3\sqrt{3}}(1, 0) \qquad \Bigg| \qquad \vec{K}' = \frac{4\pi}{3\sqrt{3}}(-1, 0)$$

$$e^{-i\vec{k}\cdot\vec{a}_1} = e^{-i2\pi/3} \qquad \Bigg| \qquad e^{-i\vec{k}\cdot\vec{a}_1} = e^{+i2\pi/3}$$

$$e^{-i\vec{k}\cdot\vec{a}_2} = e^{+i\frac{2\pi}{3}} \qquad \Bigg| \qquad e^{-i\vec{k}\cdot\vec{a}_2} = e^{-i2\pi/3}$$

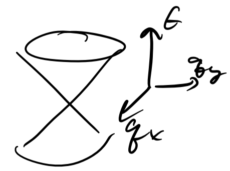
$$1 + e^{-i\vec{k}\cdot\vec{a}_1} + e^{-i\vec{k}\cdot\vec{a}_2} \overset{\vec{k}=\vec{K}+\vec{q}}{=} -\frac{3}{2}(q_x - iq_y)$$

$\cos\frac{2\pi}{3} = -1/2$

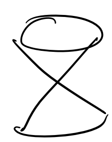
$\sin\frac{2\pi}{3} = \sqrt{3}/2$

$$H_1(k) = \begin{bmatrix} 0 & 1 + e^{-i\vec{k}\cdot\vec{a}_1} + e^{-i\vec{k}\cdot\vec{a}_2} \\ 1 + e^{i\vec{k}\cdot\vec{a}_1} + e^{i\vec{k}\cdot\vec{a}_2} & 0 \end{bmatrix}$$

$$H_1(K+q) \approx \frac{3}{2}\begin{bmatrix} 0 & -q_x + iq_y \\ -q_x - iq_y & 0 \end{bmatrix} = \frac{3}{2}(-q_x\sigma_x - q_y\sigma_y)$$



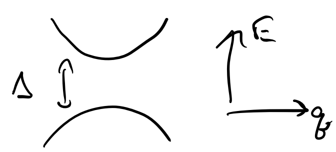
$$H_1(K'+q) \approx \frac{3}{2}\begin{bmatrix} 0 & q_x + iq_y \\ q_x - iq_y & 0 \end{bmatrix} = \frac{3}{2}(q_x\sigma_x - q_y\sigma_y)$$

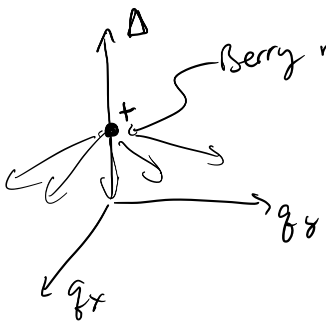


(2) Add in Δ gap opening

$$H_1(\vec{K}+q) + H_\Delta(\vec{K}+q) = -\frac{3}{2}(q_x\sigma_x + q_y\sigma_y + \Delta\sigma_z)$$

Looks like $\vec{B}\cdot\vec{S}$





Will produce a Berry flux through q_x, q_y plane.

Berry Curvature: $\Omega_{q_x q_y} = \frac{1}{2}\frac{\Delta}{(q^2+\Delta^2)^{3/2}}$

$$\Omega^{K}_{q_x q_y} = -2\,\mathrm{Im}\langle \partial_{q_x} u_{\vec{q}} | \partial_{q_y} u_{\vec{q}} \rangle$$

2 bands:

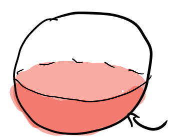
$H = \vec{d}(\vec{k})\cdot\vec{\sigma}$

$$\Omega_{k_x k_y} = \frac{1}{2}\hat{d}(\vec{k})\cdot(\partial_{k_x}\hat{d}(\vec{k}) \times \partial_{k_y}\hat{d}(\vec{k})) = \text{Area element of sphere}$$

$$\Phi = \int dq_x dq_y\, \Omega_{q_x q_y} = \frac{1}{2}\int dq_x dq_y \frac{\Delta}{(q_x^2+q_y^2+\Delta^2)^{3/2}}$$

$$= \frac{1}{2}\int_0^\infty dq \int_0^{2\pi} d\phi \frac{\Delta\, q}{(q^2+\Delta^2)^{3/2}} = \pi\Delta\left(-\frac{1}{(q^2+\Delta^2)^{1/2}}\right)\Bigg|_0^\infty = \pi = 2\pi C$$

Why? $C = 1/2$



$C_K = 1/2$ ← Chern Number

$$\Omega^{K'}_{q_x q_y} = -\frac{1}{2}\frac{\Delta}{(q^2+\Delta^2)^{3/2}} \quad (q_x \to -q_x) \Rightarrow C_{K'} = -1/2 \qquad C_{TOT} = C_K + C_{K'} = 0.$$

$\Rightarrow$ No topology

How do we switch a "Chern" number?

$$H_K = \begin{bmatrix} \Delta & \frac{3}{2}(-q_x+iq_y) \\ \frac{3}{2}(-q_x-iq_y) & -\Delta \end{bmatrix} \qquad H_{K'} = \begin{bmatrix} -\Delta & \frac{3}{2}(q_x+iq_y) \\ \frac{3}{2}(q_x-iq_y) & +\Delta \end{bmatrix}$$

$C = 1/2$ $\qquad$ $C = 1/2$

$C_{TOT} = 1$

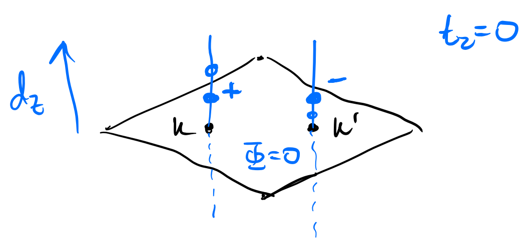
$$H_2 = 2t_2\left[\sin \vec{k}\cdot\vec{a}_2 - \sin \vec{k}\cdot\vec{a}_1 + \sin \vec{k}\cdot(\vec{a}_2-\vec{a}_1)\right]\sigma_z = \begin{cases} 3\sqrt{3}\,t_2 + O(q)^2, & \vec{k}=\vec{K}+\vec{q} \\ -3\sqrt{3}\,t_2 + O(q)^2, & \vec{k}=\vec{K}'+\vec{q} \end{cases}$$

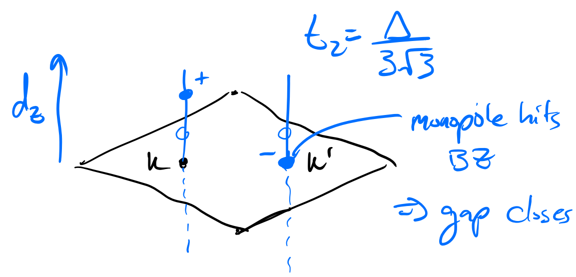
$$\Rightarrow \quad H_K = \begin{bmatrix} \Delta + 3\sqrt{3}t_2 & \frac{3}{2}(-q_x + i q_y) \\ \frac{3}{2}(-q_x - i q_y) & -(\Delta + 3\sqrt{3}t_2) \end{bmatrix}$$

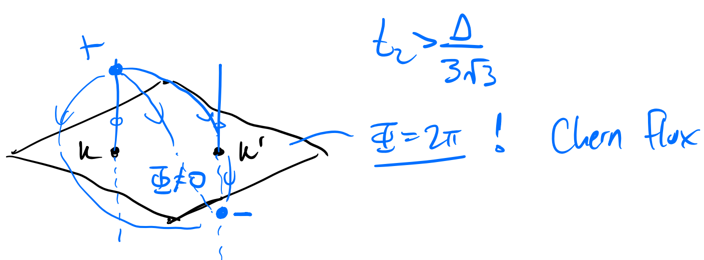
$$H_{K'} = \begin{bmatrix} \Delta - 3\sqrt{3}t_2 & \frac{3}{2}(-q_x + i q_y) \\ \frac{3}{2}(-q_x - i q_y) & -(\Delta - 3\sqrt{3}t_2) \end{bmatrix} \qquad (\Delta, t_2 > 0)$$

If $\Delta > 3\sqrt{3}t_2 \Rightarrow C=0$ trivial

If $\Delta < 3\sqrt{3}t_2 \Rightarrow C=1$ topological





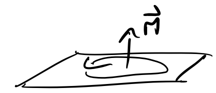


By Kubo: $\sigma_{xy} = \frac{e^2}{h}$ if $t_2 > \frac{\Delta}{3\sqrt{3}}$

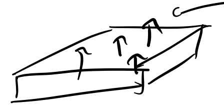
Next: Verify Numerically.

Experimentally — was difficult to see QAH

- System must be 2D
- Must be magnetic (TR broken $\perp$ to plane)
- Must be insulating — (metals more common)
- SOC must be strong (to give band inversion) — Heavy atoms needed
- Must be chemically stable



<u>One route</u>: 3D TI Bi_2Se_3 $\xrightarrow{\text{Thin film}}$



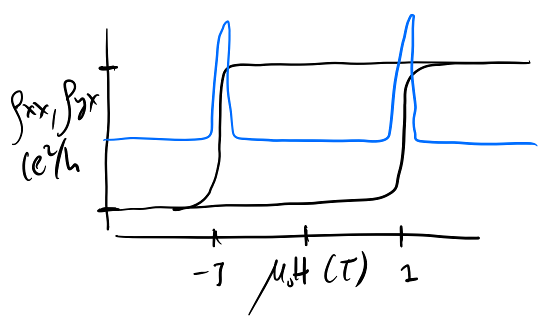
magnetic impurities

Chang (2013)

Cr-doped $(Bi, Sb)_2Te_3$

V-doped films — Chang (2015)

$(Bi_{0.29}Sb_{0.71})_{1.89}V_{0.11}Te_3$



25 mK