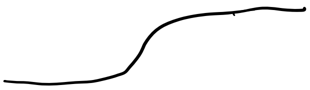
$$\frac{d\psi}{dx} + \frac{m(x)}{v_F}\sigma_z \psi = 0 \quad \text{For } \sigma_z = \pm 1 \quad \psi_\pm = e^{\mp \int_0^x \frac{m(x')}{v_F} dx'} \phi_\pm$$

$\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

Based on normalization

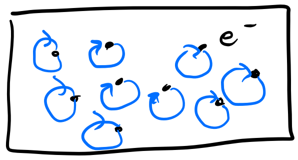


$$\Rightarrow \psi_+ = e^{-\int^x \frac{m}{v_F} dx'} \begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow \psi_- = e^{\int^x \frac{m}{v} dx'} \begin{pmatrix}0\\1\end{pmatrix}$$

Integer Quantum Hall Effect

Classical Hall Effect: Consider a 2D gas of electrons with charge e in the presence of a magnetic field $\vec{B} = B\hat{z}$. We get cyclotron motion.



$\odot \vec{B}$

Now, add an electric field $\vec{E} = E\hat{y}$. Drift velocity balances the forces:

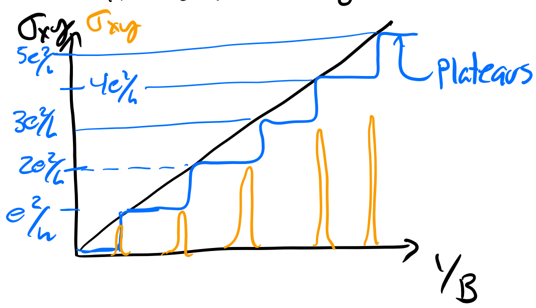
$$\vec{F} = 0 = -eE\hat{y} - e\vec{v}\times\vec{B} \Rightarrow -eE\hat{y} - e(-v_x\hat{y} + v_y\hat{x})B = 0 \qquad v_y = 0, \quad v_x = \frac{E_y}{B}$$

Get a current in the x-direction. (electrons need to "relax" to this)

$$j_x = \rho v_x = \rho\frac{E_y}{B} = \sigma_{xy} E_y \, ,$$

$\sigma_{xy} = \rho/B$ is the "Hall conductivity." This follows from Lorentz invariance, and is general.

We expect:



We can see linear dependence for small B

At high B and low T you find something different (blue).

At each plateau: $\sigma_{xy} = n\frac{e^2}{h}$, $n = 1, 2, 3, \ldots$

And σ_{xy} is quantized to many decimal places: "Integer Quantum Hall Effect"

Later, at higher magnetic fields they found plateaus at $\sigma_{xy} = \frac{p}{q}\frac{e^2}{h}$ for $p, q \in$ Integers.

"Fractional Quantum Hall Effect" — Interactions are <u>crucial</u> for FQHE. Not IQHE.

Consider non-interacting electrons in a uniform magnetic field and electric field ($\vec{B}=B\hat{z}$, $\vec{E}=E\hat{y}$). Consider only spinless electrons. One particle problem

$$H=\frac{(\vec{p}-e\vec{A})^2}{2m}-eEy$$

Work in Landau gauge $\vec{A}=-By\hat{x}$ ($\vec{\nabla}\times\vec{A}=\vec{B}$). We have

$$H=\frac{(p_x+eBy)^2}{2m}+\frac{p_y^2}{2m}-eEy$$

$[p_x,H]=0$ so all eigenstates are of the form $e^{ikx}\psi(y)$. ψ is an eigenstate of

$$H_k=\frac{(k+eBy)^2}{2m}+\frac{p_y^2}{2m}-eEy=\frac{k^2}{2m}+\frac{eBky}{m}+\frac{e^2B^2y^2}{2m}+\frac{p_y^2}{2m}-eEy$$

$$=\frac{p_y^2}{2m}+\frac{e^2B^2}{2m}(y-y_k)^2+\text{const.},\qquad y_k=\frac{mE}{eB^2}-\frac{k}{eB}$$

$$\boxed{k\frac{E}{B}-\frac{1}{2}m\left(\frac{E}{B}\right)^2}\ \star$$

Express everything in terms of

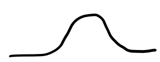
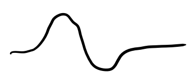
$\omega_c=\frac{eB}{m}$ cyclotron frequency

$\ell_B=\frac{1}{\sqrt{eB}}$ magnetic length

$$H_k=\frac{p_y^2}{2m}+\frac{1}{2}m\omega_c^2(y-y_k)^2,\quad y_k=y_0-k\ell^2$$

$$y_0=\frac{mE}{eB^2}$$

Harmonic Oscillator Hamiltonian!

Eigenstates are given by $\psi_n(y-y_k)$ where $\psi_0=$  $\psi_1=$ , ...

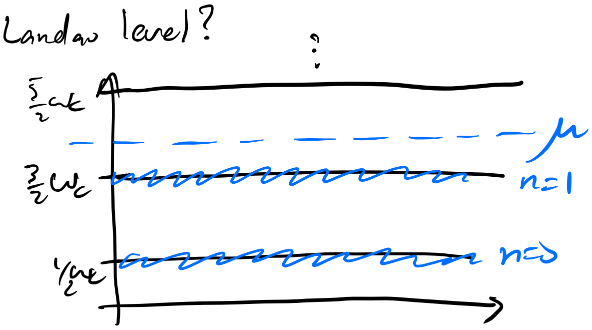
Full solution: $\psi_{k,n}(x,y)=e^{ikx}\psi_n(y-y_k)$ with energies $E_{kn}=(n+1/2)\omega_c-eEy_k+\frac{m}{2}\left(\frac{E}{B}\right)^2$

First, suppose $E=0$, then $E_{kn}=(n+1/2)\omega_c$ and all k's have the same energy.

HUGE DEGENERACY!

These are "Landau levels." How many states are in a Landau level?

Consider a finite size in the x-direction.

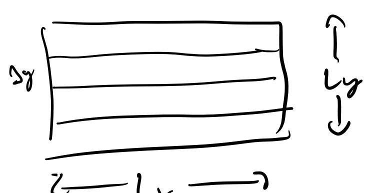


$$k=\pm\frac{2\pi}{L_x},\ \pm\frac{4\pi}{L_x},\ \ldots$$

Spacing of states: $\Delta y=y_{\frac{2(n+1)\pi}{L_x}}-y_{\frac{2n\pi}{L_x}}=\frac{2\pi}{L_x}\ell^2$

Total Number:

$$N=\frac{L_y}{\left(\frac{2\pi\ell^2}{L_x}\right)}=\frac{L_xL_y}{2\pi\ell^2}$$



This changes at the boundary, but we neglect that for now.

$$N=\frac{L_xL_y\,eB}{2\pi}=\frac{\Phi_{tot}}{2\pi/e},\qquad \frac{2\pi}{e}\ \text{is smallest quantum of flux } \Phi_0$$

$N=\Phi_{tot}/\Phi_0=$ # of flux quanta in sample.

One state per flux quantum.

Suppose chemical potential between n^{th} and $(n+1)^{th}$ L.L. Apply a weak electric field.

$$I^x_{kn} = \langle \psi_{kn} | e\frac{p_x - eA_x}{m} | \psi_{kn} \rangle \frac{1}{L_x} = \langle \psi_{kn} | e\frac{k + eBy}{m} | \psi_{kn} \rangle \frac{1}{L_x}$$

$$= \frac{1}{L_x}\left(\frac{ek}{m} + e^2\frac{B y_k}{m}\right) = \frac{e}{L_x}\left(\frac{E}{B}\right) = \frac{e}{L_x} \times (\text{drift velocity})$$

Adding up all contributions from all ψ_{kn}

$$J_x = \sum \frac{e}{L_x}\frac{E}{B} = \frac{1}{L_y}\left(n\frac{L_x L_y}{2\pi l^2}\right)\left(\frac{eE}{L_x B}\right) = n\frac{e}{2\pi l^2 B}E = n\frac{e^2}{2\pi}E$$

Putting in $\hbar$

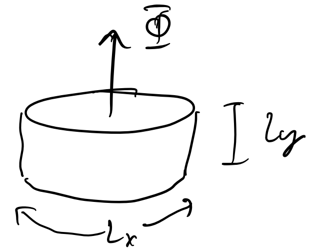
$$J_x = \frac{ne^2}{2\pi\hbar}E = \frac{ne^2}{h}E \Rightarrow \sigma_{xy} = n\frac{e^2}{h}$$

Looks like we've explained the effect but much is missing — disorder, interactions, etc.
How does this remain well-quantized?

Laughlin's Flux Argument

Consider a system w/ periodic B.C.'s in x-dir. Hamiltonian with flux Φ is:

$$H = \sum_i \left[\frac{(p_{ix} - eA_x + e\Phi/L_x)^2}{2m} + \frac{(p_{iy} - eA_y)^2}{2m} - eEy_i\right]$$



Imagine adiabatically changing Φ from 0 to $\Phi = \Phi_0 = \frac{2\pi}{e}$

Ground state will evolve into a state Ψ_Φ w/ energy E_Φ. Compare $\Delta E = E_{\Phi_0} - E_0$ in 2 different ways

1st Way Feynman-Hellman thm (1st order perturbation theory)

$$\frac{\partial E_\Phi}{\partial \Phi} = \langle \Psi_\Phi | \frac{\partial H}{\partial \Phi} | \Psi_\Phi \rangle = \langle \Psi_\Phi | \underbrace{\sum_i \frac{p_{ix} - eA_x + e\Phi/L_x}{m}\frac{e}{L_x}}_{\hat{I}_x} | \Psi_\Phi \rangle$$

$$\frac{\partial E_\Phi}{\partial \Phi} = \langle \Psi_\Phi | \hat{I}_x | \Psi_\Phi \rangle = I_\Phi$$

- Follows from Faraday's law: Flux makes an EMF $\mathcal{E} = \frac{\partial \Phi}{\partial t}$ and work is

$dW = I \cdot \mathcal{E}\,dt = I\,d\Phi \Longrightarrow dE = I\,d\Phi$ w/ a large loop I_Φ essentially Φ indep. $\Longrightarrow \Delta E = I\,\Phi_0 = I\frac{2\pi}{e}$

2^{nd} way: After inserting Φ_0, the Hamiltonian is the same up to a gauge transformation. All single particle energies are the same — Change comes from repopulating states

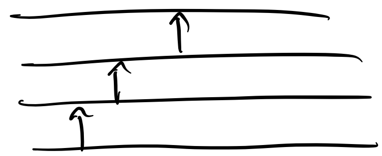
$$H_{sp} = \frac{(p_x - eA_x + e\Phi/L_x)^2}{2m} + \frac{(p_y - eA_y)^2}{2m} - eEy$$

$$= \frac{(p_x + eBy + e\Phi/L_x)^2}{2m} + \frac{p_y^2}{2m} - eEy$$

$$= \frac{[p_x + eB(y + \Phi/BL_x)]^2}{2m} + \frac{p_y^2}{2m} - eE(y + \Phi/BL_x) + \text{const.}$$

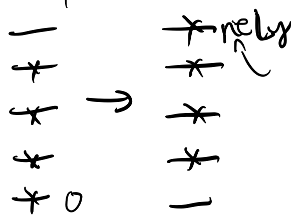
Adding Φ shifts $y \to y + \Phi/BL_x$. When $\Phi = 2\pi/e$ each orbital shifts by

$$\Delta y = \frac{2\pi}{eBL_x} = \frac{2\pi\ell^2}{L_x}$$ — Exactly the spacing between orbitals.



Each orbital shifts by 1 unit. Net result is that 1 electron per LL transferred from one edge to another

With potential difference $V = EL_y$ we have $\Delta E = neV = I\frac{2\pi}{e} \Rightarrow I = \frac{ne^2}{2\pi}V$



$\Rightarrow$ Hall conductance is $\sigma_{xy} = \frac{ne^2}{2\pi} = \frac{ne^2}{h}$

Sound familiar?

Integer Quantum Hall Effect

2D "Insulator" $\sigma_{xx} = 0$, $\sigma_{xy} \neq 0$

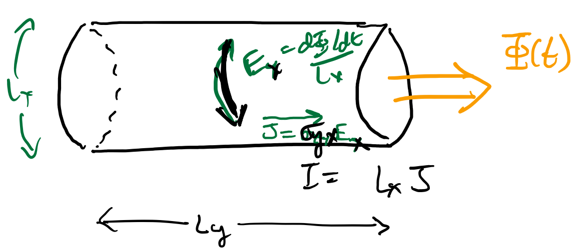
$\vec{J} = \sigma_{yx}\,\hat{z}\times\vec{E}$

$E_x = \frac{d\Phi}{dt}\frac{1}{L_x}$

$J = \sigma_{yx}\frac{d\Phi}{dt}\frac{1}{L_x}$

$I = L_x J = \sigma_{yx}\frac{d\Phi}{dt}$

Laughlin Argument



$\Phi(t=0) = 0$

$\Phi(t=T) = \phi_0 = h/e$

$\Phi = \phi_0$ can be eliminated by "large" gauge transformation in real space.

$\psi(r) \to \psi(r)e^{i\Theta(r)}$

$A_{em} \to A_{em} + \frac{\hbar}{e}\nabla_r\Theta(r)$

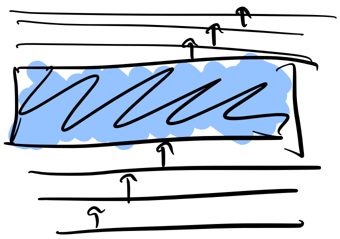
$\Phi = \oint A\cdot dr \Rightarrow \Phi + \underbrace{2\pi\hbar/e}_{\phi_0}$

$\Rightarrow H(T) = U^\dagger H(0)U$

$\Delta P = \int dt\, I(t) = \int dt\, \sigma_{xy}\frac{d\Phi}{dt} = \sigma_{xy}\frac{h}{e} \overset{\text{By large gauge}}{=} ne$

$\Rightarrow \sigma_{xy} = \frac{ne^2}{h}$

Add in (weak) interactions / disorder ($\lesssim \omega_c$). Just consider it in the bulk of the system for simplicity. — Argument immune!



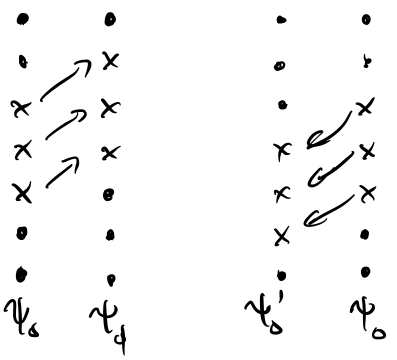
Still pump electrons — otherwise they accumulate in the bulk - which would add ω_c energy but by adiabaticity this cannot change

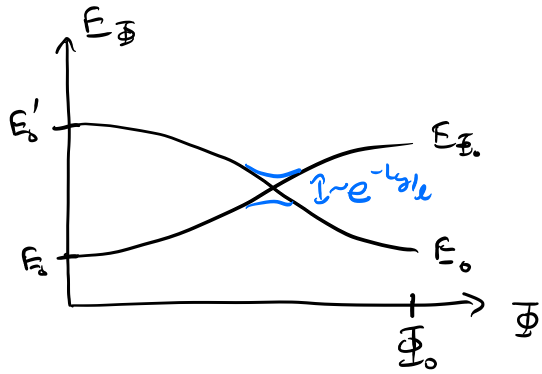
σ_{xy} robust unless we close the bulk gap!

Remarks:

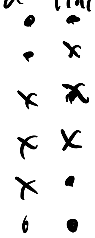
1. Why isn't $\Delta E = 0$? According to the adiabatic theorem, ground states evolve into ground states. But H is the same at beginning and end. So we must have $\Delta E = 0$.

What's going on?





For a finite size system, this will be an avoided crossing.



⟵ Differ by transferring an electron from one edge to the other ($E=0$)

$$\langle \Psi_\Phi | H | \Psi_{\Phi'} \rangle \sim \omega_c e^{-L_y/l}$$

$\Rightarrow$ Slower than ω_c, faster than $\omega_c e^{-L_y/l}$

2. Why isn't $\Delta E > \hbar\omega_c$? [Or: where does accumulated charge go?]

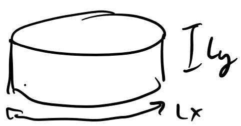
System is gapped, yet we constructed excited state Ψ_{Φ_0} w/ energy smaller than gap ($\Delta E < \hbar\omega_c$ for small V).

Answer: System is not gapped $\Longrightarrow$ gapless edge states

The IQH edge

Up until now we've ignored bdry effects. Now, we treat bdry more carefully. First suppose $E=0$

$$H = \frac{(p_x - eA_x)^2}{2m} + \frac{(p_y - eA_y)^2}{2m} + V_{edge}(y) \quad \text{where} \quad V_{edge}(y) = \begin{cases} 0, & 0 \le y \le L_y \\ \infty, & \text{else} \end{cases}$$



Landau gauge

$A_x = -By$, $A_y = 0$, $\Psi(x,y) = e^{ikx}\psi(y)$

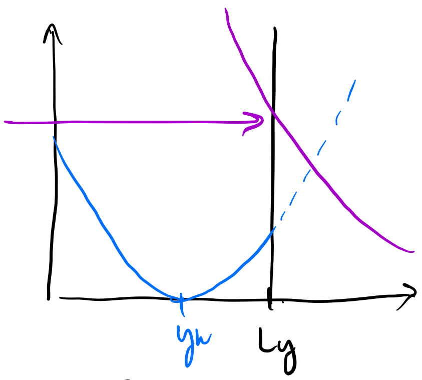
Ψ eigenstate of

$$H_k = \frac{p_y^2}{2m} + \frac{e^2 B^2}{2m}(y - y_k)^2 + V_{edge}(y) \qquad y_k = -k\ell^2$$

<u>Case 1</u>: y_k is far from the boundary (distance $\gg \ell$) $\Rightarrow$ edge has no effect and we obtain H.O. eigenstate

$$\Psi_n(0 - y_k) \approx \Psi_n(L_y - y_k) \approx 0 \implies E_{kn} \approx (n + \tfrac{1}{2})\omega_c$$

<u>Case 2</u>: y_k is close to the boundary at say $y = L_y$



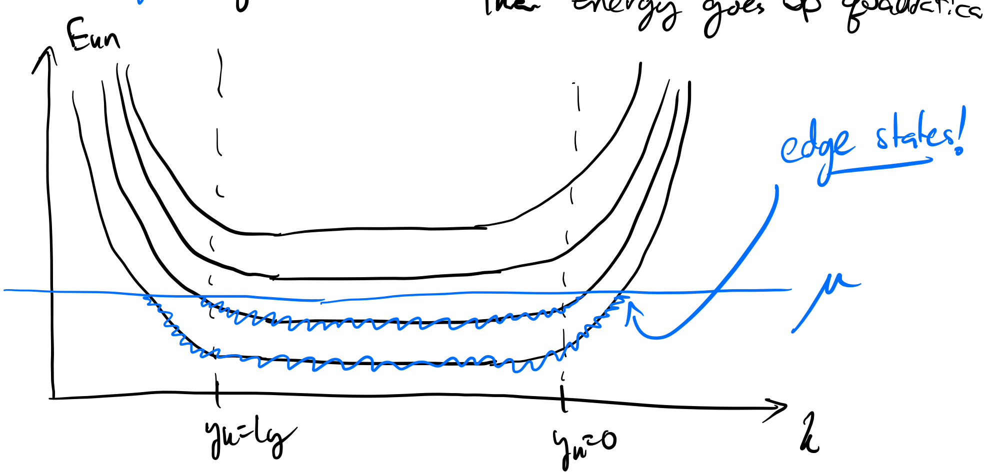
Energy $E_{kn} > (n + \tfrac{1}{2})\omega_c$

<u>Case 3</u>: y_k is outside boundary $y - L_y \gg \ell$

Estimate energy $E_{kn} \approx \frac{e^2 B^2}{2m}(y_k - L_y)^2 = \frac{e^2 B^2}{2m}(k\ell^2 + L_y)^2$

quadratic in k.

— The energy goes up quadratically w/ k for large k.



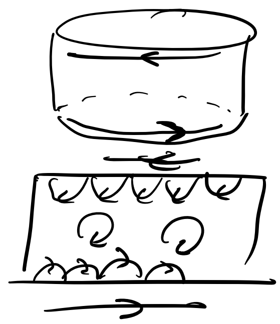
Properties of gapless excitations

1. They are <u>edge</u> excitations: Correspond to changing occupation of orbitals near the edge. These disappear w/ periodic BCs in both directions. The bulk is fully gapped.

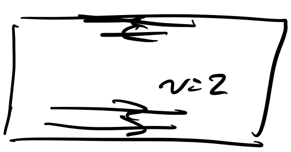
2. Edge excitations are <u>chiral</u>

   $y_k \approx 0, \quad \frac{\partial E_{nk}}{\partial k} = v > 0$ Any wave packet moves right

   $y_k \approx L_y, \quad \frac{\partial E_{nk}}{\partial k} = v < 0$

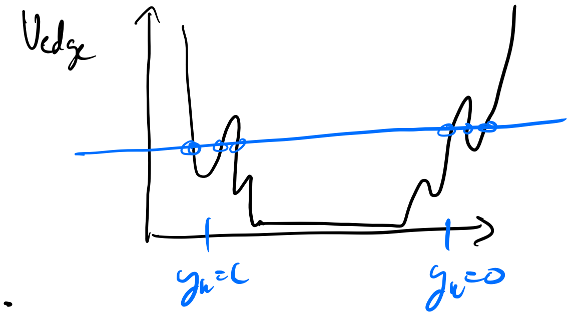


3. For n filled LL's ($\nu = n$) there are n edge modes in each direction.



4. Edge modes are protected

$\nu = 1$: We get $n_R = 2$, right moving
$n_L = 1$ left moving



$n_R - n_L = \nu \longleftarrow$ At least 1 gapless edge mode.

"Topologically protected"

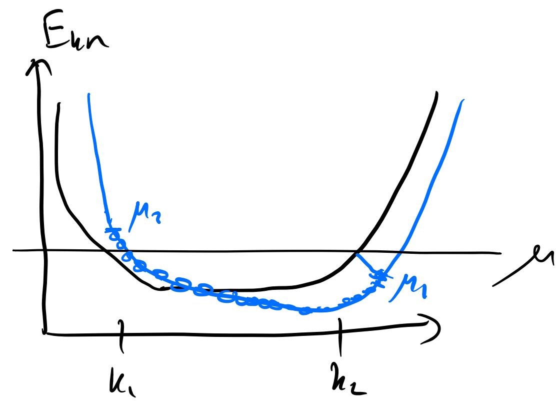
Flux threading w/ edge states $\quad \nu = n$

$$I_{kn} = \langle \psi_{kn} | \frac{1}{L_x} \frac{e}{m} (p_x - eA_x) | \psi_{kn} \rangle ,$$

$$H = \frac{(p_x - eA_x + e\Phi/L_x)^2}{2m} + \frac{(p_y - eA_y)^2}{2m} - eEy$$

$$I_{kn} = \langle \psi_{kn} | \frac{\partial H}{\partial \Phi} | \psi_{kn} \rangle = \frac{\partial E_{kn}}{\partial \Phi} = \frac{e}{L_x} \frac{\partial E_{kn}}{\partial k}$$

$$I = \sum_{kn} I_{kn} = \frac{e}{L_x} \sum_{kn} \frac{\partial E_{kn}}{\partial k}$$



Different chem. potential @ 2 edges μ_2, μ_1

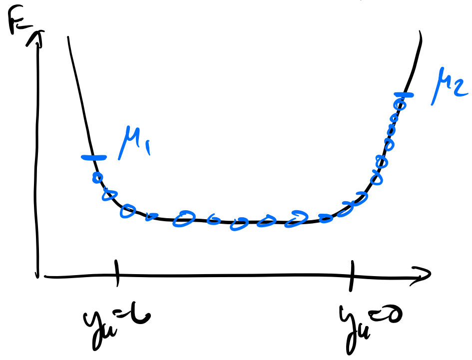
$$\mu_2 - \mu_1 \approx eELy$$

$$I = \sum_{kn} \frac{e}{L_x} \frac{\partial E_{kn}}{\partial k} = \sum_n \int_{k_1}^{k_2} \left(\frac{L_x}{2\pi} dk \right) \frac{e}{L_x} \frac{\partial E_{kn}}{\partial k} = \frac{ne}{2\pi} (\mu_1 - \mu_2)$$

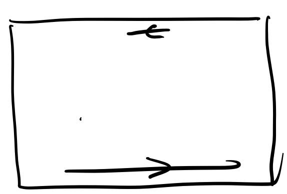
$$= \frac{-ne^2}{2\pi} \underset{\text{approx.}}{E L_y} = \frac{-ne^2}{2\pi} V_0 \qquad V_0 = EL_y = \text{electrostatic voltage}$$

$$\sigma_{xy} = \frac{ne^2}{2\pi} = \frac{ne^2}{h}$$

Better defn $V = \frac{\mu_1 - \mu_2}{e}$ (we implicitly used this)



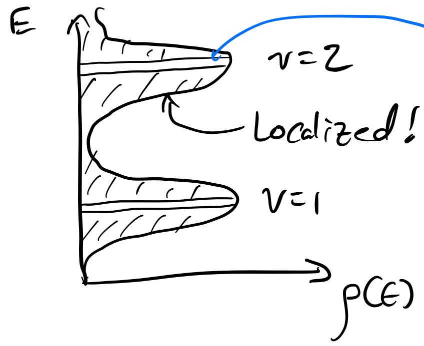
$$I_{kn} = \frac{\partial E_{kn}}{\partial k} \qquad I = \frac{e^3}{h} \left(\frac{\mu_2 - \mu_1}{e} \right)$$



Next: TKNN

Disorder Digression:

Disorder broadens bands — even LL — but $\sigma_{xy} = \frac{ne^2}{h}$ is robust — Why?

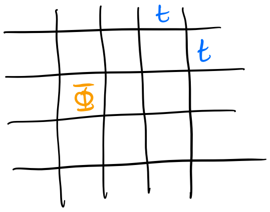


One extended state (multifractal) — Causes Hall conductivity!

Localized! (Do not contribute to conduction)

Still open questions!

TKNN — Thouless, Kohmoto, Nightingale, den Nijs (1982)



$$H = -t \sum_{\vec{r}} \left(|\vec{r}+\hat{x}a\rangle\langle\vec{r}| + |\vec{r}+\hat{y}a\rangle\langle\vec{r}| + h.c. \right)$$

Energy: $E = -2t[\cos k_x + \cos k_y]$

Add a magnetic field: $\Phi = Ba^2 = \oint_{\square} \vec{A}(\vec{r})\cdot d\vec{r}$, $A_y = Bx$

Peierls substitution:

$$|\vec{r}+\vec{R}\rangle\langle\vec{r}| \longmapsto e^{-i\frac{e}{\hbar}\int_{\vec{r}}^{\vec{r}+\vec{R}} d\vec{r}'\cdot\vec{A}(\vec{r}')} |\vec{r}+\vec{R}\rangle\langle\vec{r}|$$

$\Updownarrow$

Minimal Substitution for lattices $[\, -i\partial_x \to -i\partial_x - e\vec{A} \,]$

$$|\vec{r}+\vec{R}\rangle\langle\vec{r}| = e^{-i\vec{R}\cdot\partial_{\vec{x}}} |\vec{r}\rangle\langle\vec{r}|$$

$e^{-i}\, k$

$$e^{-i\vec{R}\cdot(\partial_x - e\vec{A})} |\vec{r}\rangle\langle\vec{r}|$$

$$e^{-iak_y} |\vec{r}\rangle\langle\vec{r}|$$

$$\Rightarrow H_B = -t\sum_{\vec{r}} \left[|\vec{r}+a\hat{x}\rangle\langle\vec{r}| + e^{-i\frac{e}{\hbar}Bxa} |\vec{r}+a\hat{y}\rangle\langle\vec{r}| + h.c. \right]$$

Translation symmetry in the y-direction.

$$H_B(k_y) = -t \sum_{x\in a\mathbb{Z}} \left[|x+a\rangle\langle x| + |x\rangle\langle x+a| + \cos\left(k_y a + \tfrac{e}{\hbar}Bxa\right) |x\rangle\langle x| \right]$$

Electric field in x-direction: $E_x = \dfrac{\partial A_x(t)}{\partial t}$

$$H_B(k_y) = -t \sum_{x\in a\mathbb{Z}} \left[e^{-i\frac{e}{\hbar}A_x(t)} |x+a\rangle\langle x| + e^{+i\frac{e}{\hbar}A_x(t)} |x\rangle\langle x+a| + \cos\left(k_y a + \underline{\tfrac{e}{\hbar}Bxa}\right) |x\rangle\langle x| \right]$$

When is this periodic?

$$\frac{eB}{\hbar}a = \frac{2\pi p}{qa}$$

$$\frac{eB}{\hbar} q a^2 = 2\pi p \Rightarrow B = \frac{2\pi\hbar}{e}\frac{p}{q}\frac{1}{a^2} = \frac{\Phi_0}{a^2}\frac{p}{q}$$

We can translate by qa in the x-direction — good unit cell! $\underline{x = n_x a}$

HW#2

$$H_B(k_x, k_y) = -t\left[\sum_{n_x=0}^{q-2} \left(e^{-i\frac{e}{\hbar}A_x} |n_x+1\rangle\langle n_x| \right) + e^{-ik_x qa} |0\rangle\langle q-1| + h.c. \right.$$
$$\left. + \sum_{n_x=0}^{q-1} \cos\left(k_y a + \tfrac{2\pi p}{q} n_x\right) |n_x\rangle\langle n_x| \right]$$

Gauge transformation: $|n_x\rangle \to e^{-ik_x n_x a} |n_x\rangle$

$$H_B(k_x,k_y) = -t \sum_{n_x=0}^{q-1} \left(e^{-i\left(\frac{e}{\hbar}A + k_x a\right)} |n_x+1\rangle\langle n_x| + \cos\left(\frac{2\pi p}{q} n_x + k_y a\right) |n_x\rangle\langle n_x| \right)$$

This has q eigenvectors & values $\epsilon_{n\vec{k}}$, $|u_{n\vec{k}}\rangle$

Recall: $J_y = \frac{dP_y}{dt} = \dot{A}\,\partial_A P_y$ ← Polarization

Note: $\partial_A = \frac{e}{\hbar}\frac{1}{a}\partial_{k_x}$

And with adiabatic pert. theory:

$$\partial_A P_y = \frac{-e}{(2\pi)^2}\int_{BZ} d^2k \; 2\,\mathrm{Im}\,\langle \partial_A u_{n\vec{k}} | \partial_{k_y} u_{n\vec{k}}\rangle$$

$$= \frac{e^2}{h}\Bigg[\underbrace{-\frac{1}{2\pi}\int_{BZ} d^2k \; 2\,\mathrm{Im}\,\langle \partial_{k_x} u_{n\vec{k}} | \partial_{k_y} u_{n\vec{k}}\rangle}_{C \in \mathbb{Z} \;\; \text{Chern!}}\Bigg]$$

Berry Curvature!

$$J_y = \frac{e^2}{h} C E_x \;\Rightarrow\; \sigma_{yx} = \frac{e^2}{h}\left[-\frac{1}{2\pi}\int_{BZ} d^2k \; \Omega_{k_x k_y}\right]$$

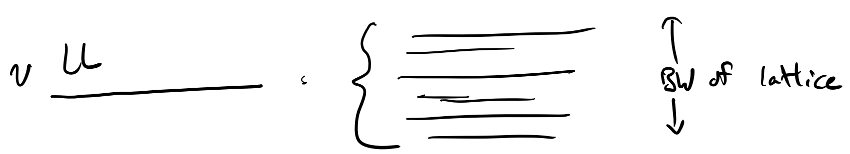
Remarks:

① These Chern numbers for gap r satisfy $\quad C_r = t_r - t_{r-1}$

$r = s_r p + t_r q$ — integer solution ⟹ Diophantine eq.

② $\sum C_r = 0$ — No net Chern number.

③ TKNN considered a LL w/ perturbation of lattice



$\sum C_r = \nu$
But all integers!

④ B can be irrational. In that case

Ⓐ Spectrum is a Cantor Set (gaps all the way down)

Ⓑ Hofstadter butterfly [w/ Chern numbers]