Rules:

- Work on this independently.
- Consulting references is OK. If they directly help you answer any part, provide the reference.
- E-mail or hand in by 12:00pm on May 12.
- Oral part of exam will be to reproduce one of the below – email me to set up that part.
- Written is 75% of the final exam grade, and oral will be 25% of the final exam grade.
- For parts labelled "**BONUS**," no partial credit will be given.
- Let me know if you suspect any part has a mistake or is not clear, so that I can correct it as quickly as possible.
- Problems vary in difficulty but not point value, that is intentional.

Problem 1. Green's function for Dirac fermions [20 pts]

Consider the single-particle Hamiltonian in k-space

$$H_0 = v_{\mathrm{F}}(\sigma_x k_x + \sigma_y k_y + m\sigma_z), \tag{1}$$

where σ_i are Pauli matrices. Note that $\psi(\mathbf{k}) = [\psi_A(\mathbf{k}), \psi_B(\mathbf{k})]^T$ are the k-space wave functions. And wherever appropriate, assume the Fermi energy is at $E = 0$.

(a) [5 pts] Write this Hamiltonian in second quantized notation using the creation and annihilations operators $c_{\mathbf{k},A}$ and $c_{\mathbf{k},B}$.

(b) [5 pts] Construct the path integral for this problem using Fermion coherent states $c_{\mathbf{k},j}\,|\psi_{\mathbf{k},j}\rangle = \psi_{\mathbf{k},j}\,|\psi_{\mathbf{k},j}\rangle$ for $j = A, B$ and $\psi_{\mathbf{k},j}$ are Grassman numbers,

$$\langle\Omega|Te^{-i\int_{-\infty}^{\infty} dt d^d x\, \hat{H}(t)}|\Omega\rangle \tag{2}$$

where $|\Omega\rangle$ the ground state of the system with chemical potential $\mu = 0$, and the full Hamiltonian is $H(t) = H_0 + f(t)\hat{V}$ for some (possibly interacting) term $\hat{V}$ and $f(t)$ adiabatically turns it on and off.

(c) [10 pts] Consider we choose $\hat{V}$ such that the Lagrangian has a source term

$$S_{\mathrm{source}} = -i\int d^2x dt \sum_{j=A,B} [\bar{\eta}_j(x)\psi_j(x) + \bar{\psi}_j(x)\eta_j(x)],$$

and integrate out the fermionic field $\psi_j(x)$ to obtain the generating function of the form

$$\exp\left(-i\int d^2k d\omega \sum_{j,j'} \bar{\eta}_{\mathbf{k},\omega,j} G_{jj'}(\mathbf{k},\omega)\eta_{\mathbf{k},\omega,j'}\right).$$

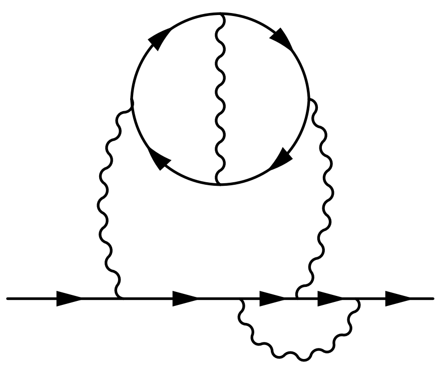
What is the Green's function $G_{jj'}(\mathbf{k},\omega)$?

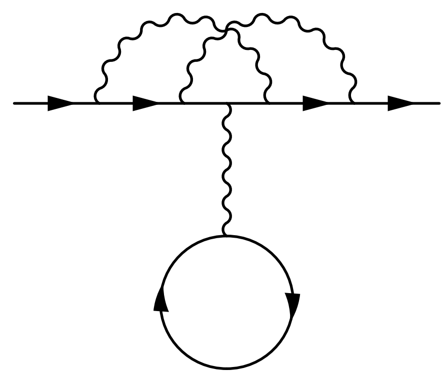
BONUS [+5 pts] Determine the poles of $G_{jj'}(\mathbf{k},\omega)$ and if they belong above or below the real axis (bonus credit only if correct imaginary part found).

Problem 2. Feynman diagrams [20 pts]

(a) [10 pts] Draw all possible topologically non-equivalent diagrams (connected and disconnected) in second-order perturbation theory with respect to the two-particle interaction $V(\mathbf{r}_1 - \mathbf{r}_2)$.

(b) [10 pts] Write down analytical expressions (in momentum space) corresponding to the following diagrams:





Problem 3. Polarization [20 pts]

Consider the usual SSH Hamiltonian in its topological phase

$$H_0 = \sum_{n=-\infty}^{\infty} (t_1 |n, A\rangle \langle n, B| + 2t_1 |n+1, A\rangle \langle n, B| + \text{h.c.}), \tag{3}$$

but with an added term that breaks the topology

$$V = W \sum_{n=-\infty}^{\infty} (|n, A\rangle \langle n, A| - |n, B\rangle \langle n, B|) \tag{4}$$

Compute the polarization in the bottom band as a function of W.

Problem 4. Two-dimensional topology [20 pts]

Consider the two-dimensional Hamiltonian

$$H_{\mathbf{k}} = \sum_{\mathbf{r}} \left[\frac{1}{2} t(\sigma_z + i\sigma_x) |\mathbf{r} - \hat{\mathbf{x}}\rangle \langle \mathbf{r}| + \frac{1}{2} t(\sigma_z + i\sigma_y) |\mathbf{r} - \hat{\mathbf{y}}\rangle \langle \mathbf{r}| + \text{h.c.} \right] - M \sum_{\mathbf{r}} \sigma_z |\mathbf{r}\rangle \langle \mathbf{r}| \quad (5)$$

where $\mathbf{r} = (n, m)$ for some integers n, m (a square lattice).

(a) [4 pts] Diagonalize the Hamiltonian in k-space, and compute its energy eigenvalues.

(b) [5 pts] Determine what values of M that close the gap and which phases are topological and their Chern numbers (set Fermi energy to $E = 0$). Draw the phase diagram.

(c) [4 pts] Assume we have a boundary at $x = 0$, derive the effective one-dimensional Hamiltonian $h(k_y)$ that describes the system (i.e., the sum $\sum_{\mathbf{r}}$ is $\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty}$ for $\mathbf{r} = (n, m)$.)

(d) [4 pts] Making the *ansatz* for the eigenstate $h(k_y) |\psi(k_y)\rangle = E(k_y) |\psi(k_y)\rangle$

$$|\psi(k_y)\rangle = \mathcal{N} \sum_{x=1}^{\infty} \lambda^{x-1} \zeta |x\rangle, \quad \zeta = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

find $\lambda = \lambda(M, k_y)$ and the energy of the state. (*Hint*: Remember that no term in the Hamiltonian has $|x = 0\rangle$, so $|0\rangle \langle 1|$ and $|1\rangle \langle 0|$ terms are identically zero – you can't hop outside of the system.) [**BONUS** +3 pts]: Derive $\zeta = (1 \quad i)^T$.

(e) [3 pts] Sketch a plot with M on the x-axis and k_y on the y-axis, indicate the regions where $|\psi(k_y)\rangle$ is a normalizeable solution. Mark the different topological phases on the x-axis.

Problem 5. Robust edge states and symmetry [20 pts]

Consider the one-dimensional, edge-state Hamiltonian

$$h(k) = vk\sigma_z + d_x(k)\sigma_x + d_y(k)\sigma_y + d_z(k)\sigma_z + \epsilon_0(k). \tag{6}$$

where the momentum k is any real number and we assume the other functions are bounded $|d_j(k)| < \Delta$, $|\epsilon_0(k)| < \Delta$ for all k. This Hamiltonian is *anomalous*: it must have come from a bulk two-dimensional Hamiltonian. This is the general form of edge states in the Quantum Spin Hall Effect.

(a) [2 pts] If $d_j = 0 = \epsilon_0$, what is the current operator J_k? What is the expectation value of J_k with respect to eigenstates of $h(k)$?

(b) [5 pts] If the system has a time-reversal symmetry with $T^2 = -1$, specifically with $T = i\sigma_y K$ in real space, what are the conditions on $d_j(k)$ and $\epsilon_0(k)$ that must be satisfied?

(c) [3 pts] What are the energy eigenvalues in general? What are they at $k = 0$, $k = \pm\infty$? Sketch a plot of E vs. k.

(d) [5 pts] Now consider the larger Hamiltonian

$$h(k) = vkI_N\sigma_z + \sum_j D^{(j)}(k)\sigma_j + E^{(0)}(k)\sigma_0, \tag{7}$$

where $D^{(j)}(k)$ and $E^0(k)$ are each $N \times N$ matrices, and I_N is the $N \times N$ identity matrix. What are all the conditions $D^{(j)}(k)$ and $E^{(0)}(k)$ must satisfy? (*Hint*: Don't forget $h(k)$ being hermitian also gives us a condition.)

(e) [5 pts] Focusing on $k = 0$, if $|\psi_E(k = 0)\rangle$ is an eigenstate with energy E show that $T|\psi_E(k = 0)\rangle$ is also a *distinct* eigenvector. What is its energy? For what values of N can a gap open? Finally, sketch the bands for $N = 2$ and $N = 3$.

BONUS [+5 pts] If Eq. (7) came from two copies of the quantum Hall effect (one with Chern number $+N$, the other with Chern number $-N$) which each individually has a $\mathbb{Z}$ topology, argue for what (e) means for the topology of the new system that has this $T^2 = -1$ symmetry.