Problem 1. Edge states in the SSH chain

In this problem we will explicitly consider both an semi-infinite chain as well as a finite chain to understand both edge states and their splitting

For the semi-infinite chain, the SSH Hamiltonian takes the form

$$H = \sum_{n=1}^{\infty} (t_1 |n, A\rangle \langle n, B| + t_2 |n+1, A\rangle \langle n, B| + \text{h.c.}). \tag{1}$$

(a) Let $|\psi\rangle = \sum_n (\psi_{n,A} |n, A\rangle + \psi_{n,B} |n, B\rangle)$ and evaluate $H|\psi\rangle = E|\psi\rangle$ to obtain a recurrence relation for $\psi_{n,A}$ and $\psi_{n,B}$.

(b) Derive a (normalizeable) solution for $E = 0$. Plot $|\psi_{n,A}|^2$ and $|\psi_{n,B}|^2$ vs. n. What conditions are needed for it to be normalizeable?

(c) Write the recurrence relations in such a way that you eliminate ψ_{nA} (to obtain a recurrence relation solely in ψ_{nB}), and using the boundary condition $\psi_{0B} = 0$, solve the recurrence relations (Recall that if $a_{n+2} = pa_{n+1} + qa_n$ that we can find two solutions with $a_n = c\lambda_1^n + b\lambda_2^n$). We require normalization so $|\psi_{nB}| \to 0$ as $n \to \infty$, this will restrict $|\lambda_j| \le 1$ — in fact you should find that for $E > 0$, $|\lambda_j| = 1$. If we let $E = 0$, can we recover the solution from part (b)? Why or why not?

(d) Lastly, consider now the finite chain

$$H = \sum_{n=1}^{L} (t_1 |n, A\rangle \langle n, B| + t_2 |n+1, A\rangle \langle n, B| + \text{h.c.}). \tag{2}$$

Argue or compute what the edge state localized at L looks like based on what you found in part (b). Approximate the $n = 1$ edge state with $|\text{Left}\rangle = \sum_{n=1}^{L} (\psi_{nA} |n, A\rangle + \psi_{nB} |nB\rangle)$ (ψ_{nA} and ψ_{nB} coming from your part (b) solution) and similarly for the edge state localized at L, which we call $|\text{Right}\rangle$. With these edge states, compute the effective Hamiltonian

$$H_{\text{edge}} = \begin{pmatrix} \langle \text{Left}|H|\text{Left}\rangle & \langle \text{Left}|H|\text{Right}\rangle \\ \langle \text{Right}|H|\text{Left}\rangle & \langle \text{Right}|H|\text{Right}\rangle \end{pmatrix}. \tag{3}$$

What are the eigenstates and what is the effective gap between them?

Problem 2. Modified Bulk SSH model

Consider the usual SSH Hamiltonian

$$H_0 = \sum_{n=-\infty}^{\infty} (t_1 \left|n, A\right\rangle \left\langle n, B\right| + t_2 \left|n+1, A\right\rangle \left\langle n, B\right| + \text{h.c.}), \tag{4}$$

but with an added term

$$V = t' \sum_{n=-\infty}^{\infty} (i \left|n+1, A\right\rangle \left\langle n, A\right| - i \left|n, A\right\rangle \left\langle n+1, A\right|) \tag{5}$$

(a) For the full Hamiltonian $H = H_0 + V$ what symmetries remain in this Hamiltonian (list the symmetries of H_0 and indicate which ones V breaks and which ones it preserves)? What is its topological classification (0, $\mathbb{Z}_2$, or $\mathbb{Z}$)?

(b) Find the k-space Hamiltonian for $H = H_0 + V$ and cast it into the form $H = \epsilon(k) + \mathbf{d}(k) \cdot \boldsymbol{\sigma}$. What are its eigenenergies?

(c) Compute the polarization of the bottom band.

Problem 3. The quantum Hall effect with spin

Electrons have spin and that directly interacts with a magnetic field. In this problem, we will explore the implications of that.

(a) First, assuming a magnetic field in the $\mathbf{B} = B\hat{\mathbf{z}}$ in a two-dimensional electron gas. Using your favorite gauge, find the eigenstates and eigenenergies of the Pauli Hamiltonian

$$H_{\mathrm{P}} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 - \frac{e}{2mc}\mathbf{B}\cdot\boldsymbol{\sigma}. \tag{6}$$

In particular, note how spin changes things: What is the degeneracy of each Landau level (make your space finite in whichever way is convenient for your gauge so that you have a total flux Φ through the system)? How has the inclusion of spin changed this?

(b) Define an operator $Q_1 = \frac{1}{\sqrt{4m}}(\mathbf{p} - \frac{e}{c}\mathbf{A})\cdot\boldsymbol{\sigma}$, and show that $H_{\mathrm{P}} = 2Q_1^2$. and show that if you have an eigenstate $|\psi_E\rangle$ then $|\psi'_E\rangle = \sqrt{2/E}\,Q_1|\psi_E\rangle$ is also an eigenstate (in general or for this specific problem). Q_1 is known as a *supercharge*.

(c) *Complex supercharge*. Let the vector potential be $\mathbf{A} = (A_x(\mathbf{r}), A_y(\mathbf{r}), 0)$ (with $\mathbf{A}$ independent of z; don't specify the gauge any further), and define

$$A = \frac{1}{\sqrt{2m}}\left[\left(p_x - \frac{e}{c}A_x\right) - i\left(p_y - \frac{e}{c}A_y\right)\right], \quad Q = (\sigma_x + i\sigma_y)A. \tag{7}$$

Compute Q^2, $\{Q, Q^\dagger\}$, and $[A, A^\dagger]$. (Recall that the cyclotron frequency is $\omega_c = \frac{eB}{mc}$.)

(d) Write the Hamiltonian in the basis of spin $\uparrow$ and spin $\downarrow$, and purely in terms of A and $A^\dagger$. Specifically, show that the partially projected operators $\langle\uparrow|H_{\mathrm{P}}|\uparrow\rangle = A^\dagger A$ and $\langle\downarrow|H_{\mathrm{P}}|\downarrow\rangle = AA^\dagger$ in terms of A and $A^\dagger$. These expressions are said to be *isospectral* in that the spectrum of both are the same except at zero energy. In particular, something called an *index theorem* relates them:

$$(\#\text{ of zeros of } A^\dagger A) - (\#\text{ of zeros of } AA^\dagger) = \Delta \tag{8}$$

What is the index Δ in this problem? (*Hint*: Don't forget the degeneracy from part (a))

(e) Given a chemical potential $\mu = \epsilon_{\mathrm{F}} = \frac{5}{2}\omega_c$ what is the magnetization $M = \langle\sigma_z\rangle = \sum_{E<\mu}\langle\psi_E|\sigma_z|\psi_E\rangle$? Relate this back to Δ from part (d).