Problem 1. Coherent state path integral

In this problem, we will derive the coherent state path integral – which is crucial for bosonic path integrals. Consider the harmonic oscillator Hamiltonian $H = \omega a^\dagger a$ where $a = \sqrt{\frac{m\omega}{2}}(x + i\frac{p}{m\omega})$. (We will neglect the zero point energy in this problem). For any complex number α, one can define a corresponding *coherent state* by

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{-\alpha a^\dagger} |0\rangle .$$

The coherent states $|\alpha\rangle$ satisfy

$$a |\alpha\rangle = \alpha |\alpha\rangle .$$

In addition, one can check that they are normalized so that

$$\langle\beta|\alpha\rangle = \exp(-|\alpha|^2/2 - |\beta|^2/2 + \beta^*\alpha), \tag{1}$$

$$1 = \int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha\rangle \langle\alpha| . \tag{2}$$

(Here $\mathrm{d}^2\alpha = \mathrm{d}(\Re[\alpha])\mathrm{d}(\Im[\alpha])$.)

(a) The coherent state time evolution operator is defined by $U(\alpha_f, t_f; \alpha_i, t_i) = \langle\alpha_f|e^{-iH(t_f - t_i)}|\alpha_i\rangle$. Writing $e^{-iH(t_f-t_i)} = (e^{-iH\Delta t})^N$ and inserting the identity operator (2) appropriately, derive an expression for U in terms of a discrete path integral over paths $\alpha(t)$. Take the continuum limit, and show that the Lagrangian is

$$L = \frac{i}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) - \omega|\alpha|^2. \tag{3}$$

(b) Show that the Lagrangian (3) is the same as the phase-space Lagrangian $L = p\dot{x} - \frac{p^2}{2m} - \frac{m\omega^2 x^2}{2}$ up to a total time derivative term.

Problem 2. Path Integral and Operator Ordering

A 3D quantum particle in a magnetic field is described by the quantum Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m}(p - A(x))^2 \\ &= \frac{1}{2m}(p^2 - pA(x) - A(x)p + A(x)^2). \end{aligned} \tag{4}$$

(Here we set $q = c = 1$ for simplicity).

(a) Writing $e^{-iH(t_f - t_i)} = (e^{-iH\Delta t})^N$, derive a discrete (Lagrangian) path integral expression for $U(x_f, t_f; x_i, t_i)$. Use the ordering of p, $A(x)$ in Eq. (4).

(b) The Hamiltonian can be equivalently written as

$$H = \frac{1}{2m}(p^2 - 2pA(x) - i\nabla \cdot A(x) + A(x)^2). \tag{5}$$

Derive a discrete (Lagrangian) path integral expression for U using the ordering in Eq. (5).

(c) Take the continuum limit and show that the first discrete integral leads to a continuum path integral with Lagrangian $L = \frac{m}{2}\dot{x}^2 + A(x)\dot{x}$, while the second leads to $L = \frac{m}{2}\dot{x}^2 + A(x)\dot{x} + \frac{i\nabla \cdot A(x)}{2m}$.

(d) The additional $\frac{i\nabla \cdot A(x)}{2m}$ term has a real physical effect (it is not a total derivative) so something must be wrong. The resolution of this apparent paradox is that continuum path integrals with terms like $A(x)\dot{x}$ are inherently ambiguous/ill-defined. Consider the following two discretizations of $A(x)\dot{x}$:

$$\left(\frac{A(x_k) + A(x_{k-1})}{2}\right)\left(\frac{x_k - x_{k-1}}{\Delta t}\right); \qquad A(x_{k-1})\left(\frac{x_k - x_{k-1}}{\Delta t}\right). \tag{6}$$

Argue that for a typical path in the path integral, the difference between these two terms is of order $(\Delta t)^0$ so that the difference between the amplitudes obtained from the two discretizations is finite in the limit $N \to \infty$. This is the path integral analogue of the operator ordering ambiguity which occurs when quantizing a classical theory.

Problem 3. Harmonic oscillator path integral

Calculate the time evolution operator $U(x_f, t_f; x_i, t_i)$ for the harmonic oscillator $H = \frac{p^2}{2m} + \frac{m\omega_0 x^2}{2}$ by generalizing the free particle calculation from class. You may wish to use the identity $\det(C_n) = \sin((n+1)x)/\sin(x)$ where C_n is the tridiagonal $n \times n$ matrix

$$C_n = \begin{pmatrix} 2\cos x & -1 & 0 & \\ -1 & 2\cos x & -1 & \dots \\ 0 & -1 & 2\cos x & \\ & \vdots & & \ddots \end{pmatrix} \tag{7}$$

Using analytic continuation, write down the imaginary time evolution operator $U_{\text{im}}(x_f, \tau_f; x_i, \tau_i)$. By examining the decay of $U_{\text{im}}(0, \tau_f; 0, \tau_i)$ in the limit $\tau_f - \tau_i \to \infty$, find the ground state energy. (*Hint:* In imaginary time $e^{-\beta H} \to |E_0\rangle \langle E_0| e^{-\beta E_0}$ as $\beta \to \infty$, and $\beta = \tau_f - \tau_i = 1/T$ for temperature T)

Problem 4. Free particle on a ring

Consider a free quantum particle on a ring. Let θ be the angular coordinate and L be the angular momentum, with $[\theta, L] = i$. The Hamiltonian is then $H = L^2/2I$.

(a) Solving the system directly, derive an expression of the partition function $\mathcal{Z} = \mathrm{tr}[e^{-\beta H}]$ of the form

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} e^{-\beta E_n}, \tag{8}$$

and compute E_n.

(b) Using the imaginary time path integral, derive an expression for $\mathcal{Z}$ of the form

$$\mathcal{Z} = A(\beta) \sum_{m=-\infty}^{\infty} e^{-F(m)/\beta}. \tag{9}$$

Calculate $A(\beta)$, $F(m)$.

(c) Compute the leading behaviour of the two expressions Eq. (8) and Eq. (9) for $\mathcal{Z}$ in the limits $\beta \to 0$, $\beta \to \infty$ and show that they agree. (The fact that they agree in general can be derived using the Poisson summation formula).